Investigating the “Artificial Gravity” of a Rotating Frame of Reference

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Abstract

Analysis of a classical mechanics problem in which a force-free mass is observed from a rotating frame of reference, leading to the appearance of “artificial gravity”.

1 Introduction

This paper addresses a Cert HE question on circular motion that asked what would happen to an apple dropped in the “artificial gravity” environment of a rotating circular space station.

In analysing the problem, we will assume that any gravitational field present is so small that it can be ignored as negligible. We will assume that the apple can be modelled as a non-rotating particle, i.e. a point mass, and we will also assume that the passenger who drops the apple remains stationary.

2 Equations of motion

Consider the diagram below, where:

ω is the angular speed of the ship (a constant)
θ_p is the angle travelled by the passenger in time t
r_s is the radius of the ship (a constant)
h is the height that the apple is dropped from, measured along the normal from the bottom of the ship
x_a is the (linear) distance travelled by the apple in time t
v is the speed of the apple once it has been dropped (constant since no forces are acting on the apple)
θ_a is the angle travelled by the apple in time t (where $0 \leqslant \theta_a < \frac{\pi}{2}$)
r_a is the (positive) radial distance of the apple from the centre of the ship

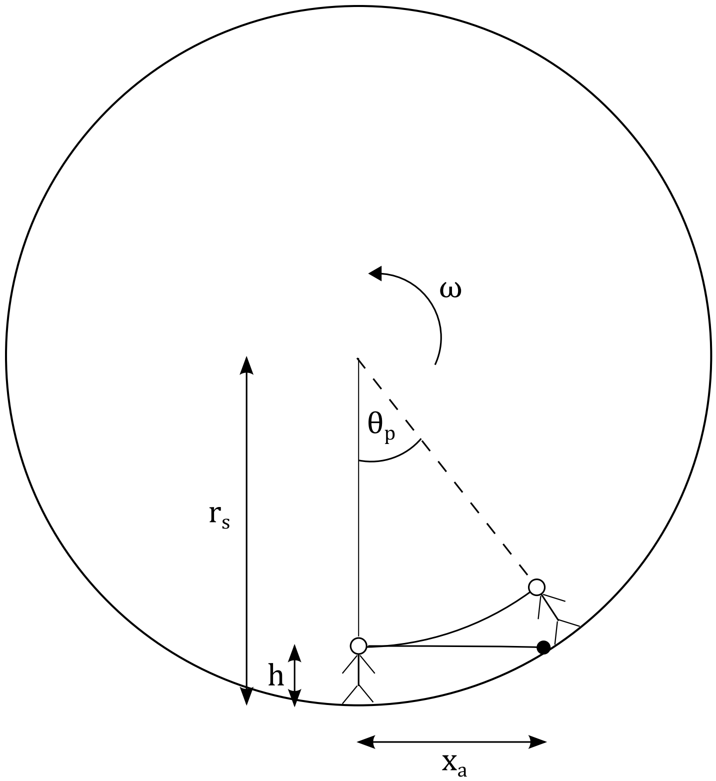
then:

$$\theta_p = \omega t \tag{1}$$

$$v = (r_s - h)\,\omega = \frac{x_a}{t}$$

$$\Rightarrow x_a = (r_s - h)\,\omega t \tag{2}$$

By simple trigonometry: $x_a = (r_s - h)\,\tan\theta_a$



But from (2):

$$x_a = (r_s - h)\,\omega t$$
$$\Rightarrow (r_s - h)\,\tan\theta_a = (r_s - h)\,\omega t$$
$$\Rightarrow \tan\theta_a = \omega t$$

$$\Rightarrow \theta_a = \arctan(\omega t) \tag{3}$$

since $0 \leqslant \theta_a < \frac{\pi}{2}$

By Pythagoras' theorem:

$$r_a^2 = x_a^2 + (r_s - h)^2 = (r_s - h)^2\,\omega^2\,t^2 + (r_s - h)^2 = (r_s - h)^2\,(1 + \omega^2\,t^2)$$

$$\Rightarrow r_a = (r_s - h)\sqrt{\omega^2\,t^2 + 1} \tag{4}$$

We will now consider the equation of motion of the apple as seen in the passenger's frame of reference using polar coordinates r and θ, where θ is measured anticlockwise from the apple's starting position.

In the passenger's frame of reference, the angle of the apple is the angle of the apple relative to the passenger, i.e.

$$\theta = \theta_a - \theta_p$$

$$\Rightarrow \theta = \arctan(\omega t) - \omega t \tag{5}$$

Since the radius of the ship is constant and since the passenger is stationary, the radial distance of the apple from the centre of the ship is no different in the passenger's frame of reference, i.e.

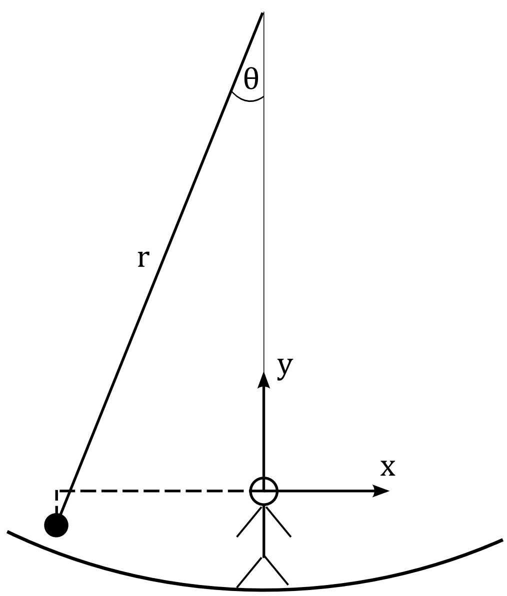
$$r = r_a$$

$$\Rightarrow r = (r_s - h)\sqrt{\omega^2 t^2 + 1} \tag{6}$$

Equations (5) and (6) are our equations of motion in polar coordinates. Since r increases monotonically as t increases and θ decreases monotonically from zero as t increases, the apple's path as seen in the passenger's frame of reference will be a spiral in the clockwise direction.

In order to plot the path of the apple, it will be useful to rewrite our polar equations of motion in terms of Cartesian coordinates, x and y. Arbitrarily, we will choose the point at which the apple is dropped as our origin, as shown in the diagram below.

Note, when the apple is in the position seen in the diagram, θ, x and y are all negative.



By simple trigonometry:

$$\cos\theta = \frac{r_s - h - y}{(r_s - h)\sqrt{\omega^2 t^2 + 1}}$$

$$\Rightarrow r_s - h - y = (r_s - h)\cos\theta\sqrt{\omega^2 t^2 + 1}$$

$$\Rightarrow y = (r_s - h)(1 - \cos\theta\sqrt{\omega^2 t^2 + 1})$$

From (5), $\theta = \arctan(\omega t) - \omega t$, so:

$$y = (r_s - h)\left(1 - \cos\left[\arctan(\omega t) - \omega t\right]\sqrt{\omega^2 t^2 + 1}\right) \tag{7}$$

Also using simple trigonometry, we see that:

$$\tan\theta = \frac{x}{r_s - h - y}$$
$$\Rightarrow x = (r_s - h - y)\tan\theta$$

Substituting $\theta = \arctan(\omega t) - \omega t$ from (5) gives:

$$\Rightarrow x = \left(r_s - h - (r_s - h)\left(1 - \cos\left[\arctan(\omega t) - \omega t\right]\sqrt{\omega^2 t^2 + 1}\right)\right)\tan\left[\arctan(\omega t) - \omega t\right]$$

$$\Rightarrow x = (r_s - h)\tan\left[\arctan(\omega t) - \omega t\right]\cos\left[\arctan(\omega t) - \omega t\right]\sqrt{\omega^2 t^2 + 1} \tag{8}$$

Using the compound angle formulae, we can write:

$$\cos\left[\arctan(\omega t) - \omega t\right] = \cos\left[\arctan(\omega t)\right]\cos(\omega t) + \sin\left[\arctan(\omega t)\right]\sin(\omega t)$$
$$\Rightarrow \cos\left[\arctan(\omega t) - \omega t\right] = \frac{\cos(\omega t)}{\sqrt{1 + \omega^2 t^2}} + \frac{\omega t\sin(\omega t)}{\sqrt{1 + \omega^2 t^2}}$$

$$\Rightarrow \cos\left[\arctan(\omega t) - \omega t\right] = \frac{\cos(\omega t) + \omega t\sin(\omega t)}{\sqrt{1 + \omega^2 t^2}} \tag{9}$$

Also:

$$\tan\left[\arctan(\omega t) - \omega t\right] = \frac{\omega t - \tan(\omega t)}{1 + \omega t\tan(\omega t)} \tag{10}$$

Substituting (9) into (7) then gives:

$$y = (r_s - h)\big(1 - \cos(\omega t) - \omega t\,\sin(\omega t)\big) \tag{11}$$

Similarly, substituting (9) and (10) into (8) and rearranging gives:

$$x = (r_s - h)\big(\omega t\,\cos(\omega t) - \sin(\omega t)\big) \tag{12}$$

Thus we have the Cartesian equations of motion for the apple as seen in the passenger's frame of reference. Using these equations, we can easily plot the x and y coordinates of the apple at any time, t.