Classical Model of a Mass on a Ramp with Coulomb Friction

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Abstract

Analysis of a classical mechanics problem involving a mass moving under gravity on a ramp exhibiting friction. The Coulomb model of dry friction is assumed in order to derive the object's equation of motion and an expression for the work done against friction by the mass.

Different shaped ramps are considered and the work done is shown to depend on the convexity of the ramp relative to the horizontal.

1 Introduction

This paper was inspired by an undergraduate exam question in which the student is asked to compare the work done against friction by a block moving down a linear ramp with that done by a block moving down a curved ramp, where the block starts from rest at some equal height in both cases and ends at some other equal height. Although it is possible to answer the question intuitively, it is interesting to analyse the problem mathematically. A general analysis allows us to determine how the shape of a ramp affects the work done against friction.

In analysing the problem, we will assume that the mass is small enough to be modelled as a particle, i.e. a point mass. We will assume that the mass can slide but does not roll or rotate. We will also assume that the ramp remains stationary and that the mass and ramp are perfectly rigid and do not deform.

First we consider the work done against friction by a particle moving in one direction between two points on a linear ramp. This turns out to depend only on the horizontal distance travelled, the particle's mass, the coefficient of friction and the gravitational field strength. The work done is independent of the particle's velocity and is independent of the ramp's slope (and therefore of the vertical distance travelled). The equation has the simple form:

$$-E = \mu mg\,|x_2 - x_1| \tag{1}$$

where $-E$ is the work done by the particle and $|x_2 - x_1|$ is the horizontal distance travelled. For a derivation, see Appendix A.

Next we consider the more general case of a ramp described by $y = f(x)$. The result of that analysis forms the content of this paper.

2 Newtonian Analysis

We wish to derive the equation of motion for a particle moving along a ramp, such that the particle does not leave the ramp's surface. This is our boundary condition.

We start from Newton's Second Law, which tells us that for a particle with constant mass m and a position given by the vector $\mathbf{r}$, the sum of the forces acting on the particle is equal to its mass multiplied by its acceleration:

$$\sum_i \mathbf{F}_i = m\ddot{\mathbf{r}}$$

We will assume that the only forces acting on the particle are its weight due to gravity ($\mathbf{W}$), the normal contact force exerted by the ramp ($\mathbf{R}$) and friction due to its contact with the ramp ($\mathbf{F}$). We will ignore such forces as air resistance, treating them as negligible. Newton's Second Law therefore gives us:

$$\mathbf{F} + \mathbf{R} + \mathbf{W} = m\ddot{\mathbf{r}} \tag{2}$$

Using Cartesian coordinates, we express the particle's position in vector notation as:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Taking the y direction as the vertical distance from the ground, and considering only ramps whose shape can be modelled in two dimensions, we set $z = 0$ and the position vector reduces to:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

We can now write Newton's Second Law as:

$$\mathbf{F} + \mathbf{R} + \mathbf{W} = m(\ddot{x}\mathbf{i} + \ddot{y}\mathbf{j})$$

where $\mathbf{W}$ is given by: $\mathbf{W} = -mg\mathbf{j}$ (assuming a constant gravitational field of magnitude g).

3 Position, Velocity and Acceleration Vectors

Since we are considering only ramps in two dimensions, we will assume that the shape of the ramp's surface can be given by the equation:

$$y = f(x)$$

where f is continuous and "well behaved" for all values of x between the start and end of the ramp.

By the condition that the particle must stay on the ramp, it is obvious that f' must exist for all values of x in our domain since it represents the gradient of the ramp. We therefore insist that f must be differentiable. As will be seen below, in order to derive the particle's equation of motion it is actually necessary for us to enforce the stronger condition that f be twice differentiable at all points in our domain.

Our boundary condition requires that the particle stay on the ramp, which means the particle's position at any point is described by the equation of the ramp. Our analysis is therefore essentially a one dimensional problem requiring only one position coordinate to uniquely specify the location of the particle on the ramp. As y is a function of x, we can eliminate y from our equations and determine the particle's equation of motion purely in terms of x, where x is of course a function of time, i.e. $x = x(t)$.

The particle's position at any point is therefore given by:

$$\mathbf{r} = x\mathbf{i} + f(x)\mathbf{j} \tag{3}$$

We can now derive expressions for the particle's velocity and acceleration at time t, by differentiating (3) with respect to t. Using the chain rule to differentiate, we find:

$$\dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{x}f'(x)\mathbf{j} \tag{4}$$

And differentiating again using the product rule and chain rule gives us:

$$\ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + (\ddot{x}f'(x) + \dot{x}^2 f''(x))\mathbf{j} \tag{5}$$

4 Coulomb Friction

In this analysis we will use the classic laws of dry friction (first discovered by Leonardo da Vinci, rediscovered by Amontons [1] and further developed by Coulomb) to model the frictional force between the particle and the ramp. These are empirical laws that hold as good approximations when the speeds involved are not too extreme [1].

When an object is at rest on a surface, Coulomb's laws of friction tell us that it experiences a frictional force, which can take any value from zero up to a maximum limit given by:

$$|\mathbf{F}| = \mu_{limiting}|\mathbf{R}|$$

in a direction opposite to the motion that the object would experience in the absence of friction [2]. When an object moves across a surface, Coulomb's laws of friction tell us that it experiences a frictional force given by:

$$|\mathbf{F}| = \mu_{kinetic}|\mathbf{R}|$$

in a direction opposite to its movement [2].

$\mu_{limiting}$ and $\mu_{kinetic}$ are constants known as the coefficient of limiting (or static) friction and the coefficient of kinetic (or dynamic) friction, respectively [2]. The value of $\mu_{limiting}$ is usually found to be greater than the value of $\mu_{kinetic}$ [1]. In this analysis however, we will simplify the equations by assuming that:

$$\mu_{limiting} = \mu_{kinetic} = \mu$$

The kinetic friction is therefore given by the equation:

$$|\mathbf{F}| = -\frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|}\mu|\mathbf{R}| \tag{6}$$

where $|\dot{\mathbf{r}}| \neq 0$ since the particle is in motion.

Modelling the static friction experienced by the particle when it is at rest is more difficult as it is no longer possible to determine the direction of the frictional force in terms of the particle's velocity vector. Instead, we will assume that the frictional force is instantaneously equal to zero when the particle is at rest. Any net resultant force on the particle in the absence of friction will then cause it to start moving. As soon as it has moved an infinitely small distance, its velocity is no longer equal to zero and the kinetic friction it experiences is given by the above expression. We therefore write that

$$|\mathbf{F}| = \mathbf{0} \tag{7}$$

when $|\dot{\mathbf{r}}| = 0$

5 Deriving the Equation of Motion

The contact force, **R**, always acts in a direction perpendicular to the tangent to the ramp at any given point. A vector along this tangent is given by:

$$\mathbf{s} = x\mathbf{i} + mx\mathbf{j} = x(\mathbf{i} + m\mathbf{j})$$

where m is the gradient at that point.

Assuming the tangent to the curve does not become vertical (at which point the particle would not technically be on the surface of the ramp), we can divide this vector by its magnitude and substitute $m = f'(x)$ to give us the unit vector along the tangent:

$$\hat{\mathbf{s}} = \frac{\mathbf{i} + f'(x)\mathbf{j}}{\sqrt{1 + f'(x)^2}}$$

R always acts in a direction 90° anticlockwise to this vector, so we can write:

$$\mathbf{R} = |\mathbf{R}|\frac{-f'(x)\mathbf{i} + \mathbf{j}}{\sqrt{1 + f'(x)^2}}$$

For the particle to remain on the ramp whilst it is in motion, its velocity must always be parallel or antiparallel to $\hat{\mathbf{s}}$ at any moment in time. Therefore:

$$\frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \pm\frac{\mathbf{i} + f'(x)\mathbf{j}}{\sqrt{1 + f'(x)^2}}$$

when $|\dot{\mathbf{r}}| \neq 0$.

By definition, $\hat{\mathbf{s}}$ has a positive horizontal component, so the plus sign in the above expression represents a velocity whose horizontal component is positive and the minus sign represents a velocity whose horizontal component is negative. While the particle is in motion, the frictional force, **F**, can therefore be expressed as:

$$\mathbf{F} = \pm\mu|\mathbf{R}|\frac{\mathbf{i} + f'(x)\mathbf{j}}{\sqrt{1 + f'(x)^2}}$$

where the sign depends on the sign of the horizontal component of the particle's velocity, $\dot{x}$.

Remembering that $\mathbf{F} = \mathbf{0}$ when $|\dot{r}| = 0$, we can write a general expression for **F** using the signum function as follows:

$$\mathbf{F} = -\frac{sgn(\dot{x})\mu|\mathbf{R}|}{\sqrt{1 + f'(x)^2}}(\mathbf{i} + f'(x)\mathbf{j}) \tag{8}$$

where we are using the convention that $sgn(x) = 0$ when $x = 0$.

We now have expressions for the particle's weight, the normal contact force and the frictional force, although the magnitude of the normal contact force, **R**, remains unknown.

Our expression of Newton's Second Law becomes:

$$-\frac{sgn(\dot{x})\mu|\mathbf{R}|}{\sqrt{1 + f'(x)^2}}(\mathbf{i} + f'(x)\mathbf{j}) + \frac{-|\mathbf{R}|}{\sqrt{1 + f'(x)^2}}(f'(x)\mathbf{i} - \mathbf{j}) - mg\mathbf{j} = m\ddot{x}\mathbf{i} + m(\ddot{x}f'(x) + \dot{x}^2 f''(x))\mathbf{j} \tag{9}$$

We can split this vector equation into its horizontal and vertical components to give us two simultaneous equations both involving $|\mathbf{R}|$. We then eliminate $|\mathbf{R}|$ from the equations and note that the mass, m, also cancels out, meaning that the particle's motion is independent of its mass.

Finally, after rearranging and simplifying, we arrive at our equation of motion for the particle:

$$\ddot{x} = \frac{-(g + \dot{x}^2 f''(x))(sgn(\dot{x})\mu + f'(x))}{1 + f'(x)^2} \tag{10}$$

For the full derivation of this equation, see Appendix B.

6 Work Done against Friction

The work done, E, by a force, $\mathbf{F}$, is defined as the line integral:

$$E = \int_c \mathbf{F} \cdot d\mathbf{r}$$

where the integral is evaluated over the path traversed, C.

From Newton's Third Law we can see that the work done *against* friction by the particle will be equal to minus the work done *by* friction on the particle, so our expression for the work done against friction will be equal to $-E$

As the particle in our system can retraverse its path, the general expression for the work done will be better expressed as an integral over time. We therefore rewrite the above definition as:

$$-E = -\int_{t_1}^{t_2} \mathbf{F} \cdot \dot{\mathbf{r}}\, dt$$

Our definition of the frictional force $\mathbf{F}$ given in (8) depends on $|\mathbf{R}|$, so to find the work done we must first derive an expression for $|\mathbf{R}|$. Using a process similar to that from which we derived the equation of motion, we start from (9), split the vector equation into components to form two simultaneous equations and then rearrange to find:

$$|\mathbf{R}| = \frac{m(g + \dot{x}^2 f''(x))}{\sqrt{1 + f'(x)^2}} \tag{11}$$

For the full derivation of this equation, see Appendix C.

Putting this expression for $|\mathbf{R}|$ into (8), we get:

$$\mathbf{F} = -\frac{sgn(\dot{x})\mu m(g + \dot{x}^2 f''(x))}{1 + f'(x)^2}(\mathbf{i} + f'(x)\mathbf{j}) \tag{12}$$

Taking our expression for $\dot{\mathbf{r}}$ from (4), we take the dot product with the above expression for $\mathbf{F}$ to express the integral for the work done over time as:

$$-E = \mu m \int_{t_1}^{t_2} sgn(\dot{x})(g + \dot{x}^2 f''(x))\, \dot{x}\, dt \tag{13}$$

This can also be expressed as:

$$-E = \mu mg \int_{t_1}^{t_2} |\dot{x}|\ dt + \mu m \int_{t_1}^{t_2} |\dot{x}^3| f''(x)\ dt \tag{14}$$

Thus we have derived an expression for the total work done against friction by the particle between times t_1 and t_2 .

We now consider the amount of work done by the particle when it moves from one point to another without retraversing its path. Clearly, this means that the sign of $\dot{x}$ does not change. During this motion we can consider the time, t, as a function of x as there is a one-to-one mapping between them.

Taking $sgn(\dot{x})$ as constant and changing the variable from t to x in (13) we write:

$$-E = \mu m\ sgn(\dot{x}) \int_{x_1}^{x_2} (g + \dot{x}^2 f''(x))\ dx \tag{15}$$

Which simplifies to:

$$-E = \mu mg\ |x_2 - x_1| + \mu m\ sgn(\dot{x}) \int_{x_1}^{x_2} \dot{x}^2 f''(x)\ dx \tag{16}$$

Notice that the first term is identical to (1), i.e. the work done by a particle moving along a linear ramp. This important result tells us that the work done moving along a curved ramp depends on the convexity of the curve, i.e. $f''(x)$.

When the ramp is convex downward between x_1 and x_2, $f''(x)$ is positive and the work done is therefore greater than it would be for a linear ramp between the same two points.

When the ramp is concave downward between x_1 and x_2, $f''(x)$ is negative and the work done is less than it would be for a linear ramp between the same two points.

The convexity of the ramp could change sign between x_1 and x_2, in which case the work done may be less than, greater than or even equal to that done on an equivalent linear ramp.

It is therefore clear that the work done against friction on a curved ramp does not solely depend on the length of the path travelled, but on the shape of the curve and the speed at which the particle is travelling.

7 Conservation of Energy

Although (16) is instructive in illustrating the relationship between the work done against friction and the convexity of the ramp, a more useful approach to calculating the work done against friction is to use the law of conservation of energy. The work done against friction is simply equal to the loss in mechanical energy from the system. We express this using the following equation:

$$\Delta U + \Delta K - E = 0$$

or

$$-E = -\Delta U - \Delta K \tag{17}$$

where ΔU is the change in gravitational potential energy and ΔK is the change in kinetic energy.

Considering the work done between times t_1 and t_2, this becomes:

$$-E = -mg[f(x_2) - f(x_1)] - \frac{1}{2}m(|\dot{\mathbf{r}}_2|^2 - |\dot{\mathbf{r}}_1|^2) \tag{18}$$

Using (4), this can be expressed purely in terms of x and $\dot{x}$ at times t_1 and t_2:

$$-E = mg[f(x_1) - f(x_2)] + \frac{1}{2}m[(\dot{x}_1)^2\,(1 + f'(x_1)^2) - (\dot{x}_2)^2\,(1 + f'(x_2)^2] \tag{19}$$

Thus we can easily calculate the work done against friction at any time given the particle's initial and final conditions.

8 Numerical Solution

We now consider a numerical approximation method for solving the equation of motion iteratively from any given initial conditions.

Our equation of motion (10) is a non-linear second order differential equation. To simplify matters we will first convert it into a pair of simultaneous first order equations by making a simple substitution for $\dot{x}$, which gives us:

$$\begin{cases} v = \dot{x} & (20) \\ \dot{v} = \dfrac{-(g + v^2 f''(x))(sgn(v)\mu + f'(x))}{1 + f'(x)^2} & (21) \end{cases}$$

We can now use Euler's method to derive an iterative expression from which to calculate the approximate values of v and $\dot{v}$ at a time $t + \delta t$, given their values at time t.

Although Euler's method is not in general the fastest converging approximation algorithm, it is the simplest and will therefore serve to illustrate the general approach we take in solving a problem of this nature.

One starts by considering the "first principles" definition of a derivative (also known as Newton's difference quotient). This is the ratio of two infinitely small changes in related variables. This gives us:

$$v(t) \simeq \frac{x(t + \delta t) - x(t)}{\delta t}$$

which rearranges to:

$$x(t + \delta t) \simeq x(t) + v(t)\,\delta t$$

and similarly:

$$v(t + \delta t) \simeq v(t) + \dot{v}(t)\,\delta t$$

Substituting back from (20) and (21) gives us our desired iterative equations:

$$\begin{cases} x(t + \delta t) \simeq x(t) + \dot{x}(t)\,\delta t & (22) \\ \dot{x}(t + \delta t) \simeq \dot{x}(t) + \dfrac{-\big[g + \dot{x}(t)^2 f''\big(x(t)\big)\big]\big[sgn\big(\dot{x}(t)\big)\mu + f'\big(x(t)\big)\big]}{1 + f'\big(x(t)\big)^2}\,\delta t & (23) \end{cases}$$

Using (22) and (23) we can compute all values of x and $\dot{x}$ to any degree of accuracy, given their values at some initial time. The smaller the value we choose for δt, the closer our approximation will approach the exact solution. This is ideally done by a computer program, which can then be used to plot y against x (the particle's spatial coordinates) or $\dot{x}$ against x (the particle's phase space coordinates) in real time to simulate our particle's behaviour.

Now that we have an iterative solution for x and $\dot{x}$ at time t, we can also calculate the work done against friction using (19).

Finally, we consider the condition for the particle to remain on the ramp. Our equation of motion and its solution are only valid whilst the particle is actually on the ramp. However, with enough velocity, it is obvious that the particle may leave the surface of a non-linear ramp. We now derive the condition under which this occurs.

Although $\mathbf{R} = \mathbf{0}$ when the particle leaves the ramp, we will not consider this alone as the condition for the particle leaving the ramp, since the particle can still be infinitely close to the ramp as the normal force approaches zero. Instead, we will compare the particle's path along the ramp between two points very close together, with the path it would take if the normal force were equal to zero.

If the particle were to leave the ramp, we could calculate its vertical motion using the "suvat" equation: $s = ut + \frac{1}{2}at^2$

Considering a very small change in the particle's vertical displacement, we see that this becomes:

$$\delta y = \dot{y}(t)\delta t - \frac{1}{2}g\,\delta t^2$$

Using the chain rule to write $\dot{y}$ as $\dot{x}f'(x)$, we arrive at:

$$\delta y = f'(x)\dot{x}\,\delta t - \frac{g\,\delta t^2}{2} \tag{24}$$

We wish to compare this to the change of vertical displacement the particle would experience if it remains on the ramp. This is given by the equation defining the ramp:

$$\delta y = f(x + \delta x) - f(x)$$

which we can rewrite, expressing x as a function of t:

$$\delta y = f\big(x(t + \delta t)\big) - f\big(x(t)\big) \tag{25}$$

So our condition for the particle to remain on the ramp is:

$$f\big(x(t + \delta t)\big) - f\big(x(t)\big) \le f'\big(x(t)\big)\,\dot{x}\,\delta t - \frac{g\,\delta t^2}{2} \tag{26}$$

A Derivation of work done by particle on a linear ramp

We wish to derive an expression for the work done against friction by a particle moving on a linear ramp between two points, $\mathbf{r}_1$ and $\mathbf{r}_2$, during which time the particle's velocity does not change direction.

The component of weight down a linear slope is given by: $|\mathbf{W}|sin\theta$, where θ is the angle at which the ramp is inclined with respect to the horizontal.

By the condition that the particle not leave the ramp, the normal contact force must be of equal magnitude to the component of weight perpendicular to the ramp (assuming no other forces, e.g. air resistance, are acting on the particle), so:

$$\mathbf{R} = -\mathbf{W}cos\theta$$

Coulomb's laws of friction tell us that when an object is moving, the friction acting on it is:

$$|\mathbf{F}| = \mu|\mathbf{R}| = \mu|\mathbf{W}|cos\theta$$

where μ is the coefficient of kinetic friction [2].

Since $\mathbf{W}$ is constant, $\mathbf{R}$ is constant and hence $\mathbf{F}$ is also constant. The Work done by a constant force moving in a straight line is given by: $\mathbf{F} \cdot (\mathbf{r}_2 - \mathbf{r}_1)$

Now, friction always acts opposite to the direction of motion, so the work done by friction is given by:

$$E = -\mu|\mathbf{W}||\mathbf{r}_2 - \mathbf{r}_1|cos\theta$$

where $|\mathbf{r}_2 - \mathbf{r}_1|$ is the distance travelled along the ramp. By simple trigonometry, this is equal to:

$$\frac{|x_2 - x_1|}{cos\theta}$$

where x is taken to be the horizontal coordinate.

This gives us the expression for the work done by friction as:

$$E = -\mu|\mathbf{W}||x_2 - x_1|$$

Since $|\mathbf{W}| = mg$ and since the work done by friction is equal and opposite to the work done against friction, we arrive at:

$$-E = \mu mg\,|x_2 - x_1|$$

Q.E.D.

B Derivation of the general equation of motion

Starting from (9):

$$-\frac{sgn(\dot{x})\mu|\mathbf{R}|}{\sqrt{1+f'(x)^2}}(\mathbf{i}+f'(x)\mathbf{j})+\frac{-|\mathbf{R}|}{\sqrt{1+f'(x)^2}}(f'(x)\mathbf{i}-\mathbf{j})-mg\mathbf{j}=m\ddot{x}\mathbf{i}+m(\ddot{x}f'(x)+\dot{x}^2f''(x))\mathbf{j}$$

$$\Rightarrow \begin{cases} -sgn(\dot{x})\mu|\mathbf{R}|-f'(x)|\mathbf{R}|=m\ddot{x}\sqrt{1+f'(x)^2} \\ -sgn(\dot{x})\mu|\mathbf{R}|f'(x)+|\mathbf{R}|=m(\ddot{x}f'(x)+\dot{x}^2f''(x))\sqrt{1+f'(x)^2}+mg\sqrt{1+f'(x)^2} \end{cases}$$

$$\Rightarrow \begin{cases} |\mathbf{R}|[f'(x)+sgn(\dot{x})\mu]=-m\ddot{x}\sqrt{1+f'(x)^2} \\ |\mathbf{R}|[1-sgn(\dot{x})\mu f'(x)]=m(\ddot{x}f'(x)+\dot{x}^2f''(x)+g)\sqrt{1+f'(x)^2} \end{cases}$$

$$\Rightarrow \begin{cases} |\mathbf{R}|[1-sgn(\dot{x})\mu f'(x)][f'(x)+sgn(\dot{x})\mu]=[1-sgn(\dot{x})\mu f'(x)][-m\ddot{x}\sqrt{1+f'(x)^2}] \\ |\mathbf{R}|[1-sgn(\dot{x})\mu f'(x)][f'(x)+sgn(\dot{x})\mu]=[f'(x)+sgn(\dot{x})\mu][m(\ddot{x}f'(x)+\dot{x}^2f''(x)+g)\sqrt{1+f'(x)^2}] \end{cases}$$

Equating these two expressions gives:

$$[1-sgn(\dot{x})\mu f'(x)][-m\ddot{x}\sqrt{1+f'(x)^2}]=[f'(x)+sgn(\dot{x})\mu][m(\ddot{x}f'(x)+\dot{x}^2f''(x)+g)\sqrt{1+f'(x)^2}]$$

$$\Rightarrow [1-sgn(\dot{x})\mu f'(x)][-m\ddot{x}]=[f'(x)+sgn(\dot{x})\mu][m(\ddot{x}f'(x)+\dot{x}^2f''(x)+g)]$$

$$\Rightarrow [sgn(\dot{x})\mu f'(x)-1]\ddot{x}=[f'(x)+sgn(\dot{x})\mu](\ddot{x}f'(x)+\dot{x}^2f''(x)+g)$$

$$\Rightarrow [sgn(\dot{x})\mu f'(x)-1]\ddot{x}-[f'(x)+sgn(\dot{x})\mu]\ddot{x}f'(x)=[f'(x)+sgn(\dot{x})\mu](\dot{x}^2f''(x)+g)$$

$$\Rightarrow\ -\ddot{x}-f'(x)^2\ddot{x}=[f'(x)+sgn(\dot{x})\mu](\dot{x}^2f''(x)+g)$$

$$\Rightarrow \ddot{x}[1+f'(x)^2]=-[f'(x)+sgn(\dot{x})\mu](\dot{x}^2f''(x)+g)$$

$$\Rightarrow \ddot{x}=\frac{-(g+\dot{x}^2f''(x))(sgn(\dot{x})\mu+f'(x))}{1+f'(x)^2}$$

Q.E.D.

C Derivation of the magnitude of the normal contact force

Starting from (9):

$$-\frac{sgn(\dot{x})\mu|\mathbf{R}|}{\sqrt{1+f'(x)^2}}(\mathbf{i}+f'(x)\mathbf{j})+\frac{-|\mathbf{R}|}{\sqrt{1+f'(x)^2}}(f'(x)\mathbf{i}-\mathbf{j})-mg\mathbf{j}=m\ddot{x}\mathbf{i}+m(\ddot{x}f'(x)+\dot{x}^2f''(x))\mathbf{j}$$

$$\Rightarrow \begin{cases} -sgn(\dot{x})\mu|\mathbf{R}|-f'(x)|\mathbf{R}|=m\ddot{x}\sqrt{1+f'(x)^2} \\ -sgn(\dot{x})\mu|\mathbf{R}|f'(x)+|\mathbf{R}|=m(\ddot{x}f'(x)+\dot{x}^2f''(x))\sqrt{1+f'(x)^2}+mg\sqrt{1+f'(x)^2} \end{cases}$$

$$\Rightarrow \begin{cases} |\mathbf{R}|[-f'(x)-sgn(\dot{x})\mu]=m\ddot{x}\sqrt{1+f'(x)^2} \\ |\mathbf{R}|[1-sgn(\dot{x})\mu f'(x)]=m(\ddot{x}f'(x)+\dot{x}^2f''(x)+g)\sqrt{1+f'(x)^2} \end{cases}$$

$$\Rightarrow \begin{cases} |\mathbf{R}|[f'(x)^2+sgn(\dot{x})\mu f'(x)]=-mf'(x)\ddot{x}\sqrt{1+f'(x)^2} \\ |\mathbf{R}|[1-sgn(\dot{x})\mu f'(x)]=m(\ddot{x}f'(x)+\dot{x}^2f''(x)+g)\sqrt{1+f'(x)^2} \end{cases}$$

Adding these two equations together gives:

$$|\mathbf{R}|(f'(x)^2+1)=m(\dot{x}^2f''(x)+g)\sqrt{1+f'(x)^2}$$

$$\Rightarrow |\mathbf{R}|=\frac{m(g+\dot{x}^2f''(x))}{\sqrt{1+f'(x)^2}}$$

Q.E.D.

References

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[2] Cullerne et al. *The Penguin Dictionary of Physics* (3rd ed) (London: Penguin Books, 2000), p. 169