Introducing Chaos

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"Let us imagine an Intelligence who would know at a given instant of time all forces acting in nature and the position of all things of which the world consists; let us assume, further, that this Intelligence would be capable of subjecting all these data to mathematical analysis. Then it could derive a result that would embrace in one and the same formula the motion of the largest bodies in the universe and of the lightest atoms. Nothing would be uncertain for this Intelligence. The past and the future would be present to its eyes." (Laplace, Analytic Theory of Probabilities, 1812)

Introduction

In this paper I present one of the most fashionable and exciting areas of physics: **chaos theory**. As an example, I will discuss the series L-R-varactor circuit – a fairly straightforward, physical system by which to illustrate the phenomenon of chaos.

Of course, chaos is not only observed by physicists. What makes it so fundamental, in fact, is that it is a feature of many different types of nonlinear system: it is found in chemistry, biology, ecology and even economics. One of the first scientists to study chaos in a dynamical system was Edward Lorenz – a meteorologist!

I am sure that many readers will have heard of chaos – probably from watching Jurassic Park – but I believe a definition is called for.

What is Chaos?

Chaos, in the scientific sense of the word, should perhaps be more accurately called deterministic chaos, since it is <u>not</u> disorder; rather it is a form of order without periodicity. This leads me to a qualitative definition of chaos as:

Chaos

Bounded, aperiodic, pseudorandom behaviour of nonlinear deterministic systems which exhibit sensitive dependence on initial conditions.

The last part of this definition highlights an important difference between linear and chaotic systems: whilst an uncertainty in the initial conditions of a linear system grows linearly in time, an uncertainty in the initial conditions of a chaotic system grows exponentially in time. Because of its disastrous consequences for long-term weather prediction, sensitivity on initial conditions has become popularly known as the **Butterfly Effect** – so named after Lorenz contemplated the question: "does the flap of a butterfly's wings in Brazil set off a tornado in Texas?" It is interesting to note here that it could alternatively have become known as the Seagull Effect after one of Lorenz's earlier metaphors! (Gleick, 1988, p.322)

I'd now like to move on to take a more quantitative look at chaos. We shall be looking at dynamical systems, i.e. systems that vary continuously with time. Such systems can always be expressed as a set of n coupled 1st order ordinary differential equations of the form:

$$\dot{x}_1 = f_1(x_1,\, x_2, \dots ,x_n)$$

$$\dot{x}_2 = f_2(x_1,\, x_2, \dots ,x_n)$$

$$\vdots$$

$$\dot{x}_i = f_i(x_1,\, x_2, \dots ,x_n)$$

$$\vdots$$

$$\dot{x}_n = f_n(x_1,\, x_2, \dots ,x_n)$$

where the x_i are the independent dynamical variables of the system. In classical mechanics, for example, the x_i are the position and momenta variables. These equations are called autonomous because the functions f_i are not explicitly dependent on time. They can be written more concisely using vector notation:

$$\underline{\dot{x}} = \underline{f}(\underline{x})$$

The vector function $\underline{f}$ is called a **flow**.

Any such system is completely deterministic because the variables only implicitly depend on time: we can imagine time as being like a parameter. The solution to these equations is a curve in n-dimensional space where the axes are the independent variables and $\underline{x}$ is the position vector of a point on the curve. This space is a very useful way of looking at dynamical systems and is known as **phase space** or state space. A curve in phase space is called a **trajectory**.

A central theorem relating to phase space is:

The No-Intersection Theorem

Two distinct phase space trajectories cannot intersect (in a finite period of time). Nor can a single trajectory cross itself at a later time.

This is based on the notion of determinism; any intersecting of trajectories would imply that the system is not deterministic, since past and future states of the system at such an intersection would not be uniquely defined.

Having introduced you to the idea of phase space, we can now look at the necessary conditions for chaotic motion.

Necessary conditions for chaos in a dynamical system

(I) The system has at least three independent variables.

(II) The equations of motion contain at least one nonlinear term that couples several of the variables.

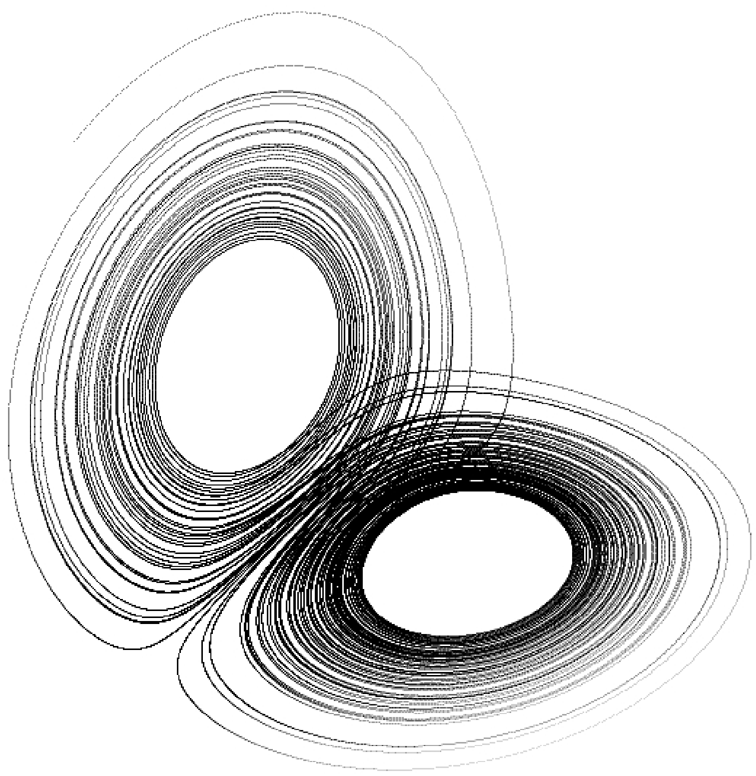
The first condition allows for:

(a) Divergence of trajectories
(b) Confinement of the motion to a finite region of phase space
(c) Uniqueness of the trajectory

This is not possible in one or two dimensions.

The second condition allows for the aperiodicity which is required for chaotic motion.

Whilst these conditions do not imply that a particular system will be chaotic, they do allow for the possibility. A very well-known example of chaos in three dimensional phase space is the Lorenz Attractor:



An **attractor** is the name given to the point, or points, to which a trajectory is 'attracted'. An important question is 'how do we know if a system's behaviour is actually chaotic?' The answer to this will lead me to a formal, quantitative definition of chaos.

You'll remember I mentioned that, in a chaotic system, two initial points will diverge exponentially. This turns out to be just the property we need to understand chaos quantitatively. If we imagine a small, rectangular volume of differing initial conditions in a 3d phase space of sides s_1, s_2 and s_3, we would expect that volume to evolve as:

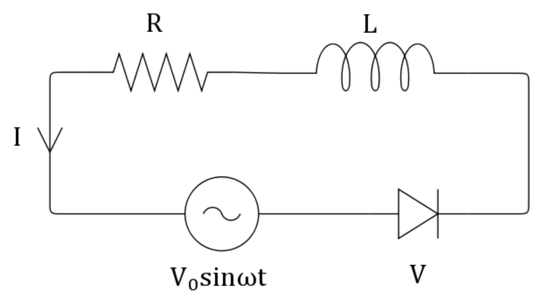
$$V(t) = \left(s_1 e^{\lambda_1 t}\right)\left(s_2 e^{\lambda_2 t}\right)\left(s_3 e^{\lambda_3 t}\right) = s_1\, s_2\, s_3\, e^{(\lambda_1+\lambda_2+\lambda_3)t}$$

Where each $s_i e^{\lambda it}$ factor is the **convergence** or **divergence** in one of the three dimensions. The values of the λ_i will vary along the trajectory, but we need to consider their average values. The λ_i are called **Lyapunov exponents** and their average values can be calculated numerically using a computer.

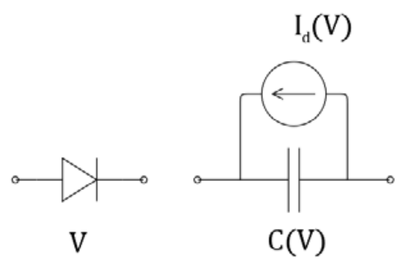
We can now define a chaotic trajectory as one for which there exists at least one average Lyapunov exponent that is positive. This definition also holds for phase spaces with higher numbers of dimensions than three. For an n-dimensional phase space, there are n Lyapunov exponents.

I think it's now time to stop talking in general terms and take a look at a real physical situation in which chaos occurs.

The L-R-varactor series circuit



The varactor diode can be modelled as an ideal diode in parallel with a nonlinear capacitance:



I_d is given by the **ideal diode equation**:

$$I_d(V) = I_0(e^{V/\phi} - 1)$$

The capacitance is taken as the sum of the junction capacitance and the storage capacitance:

$$C(V) = C_j + C_s$$

where:

$$C_j(V) = C_{j0}\left(1 - \frac{V}{\Phi}\right)^{-\gamma}$$
$$C_s(V) = C_{s0}\, \mathrm{e}^{V/\Phi}$$

ϕ is known as the **thermal voltage** and is given by:

$$\phi \equiv \frac{k\mathrm{T}}{e}$$

where k is Boltzmann's constant, e is the electron charge, and T is temperature.

Φ and γ are properties of the varactor, known as the **potential barrier** and **grading coefficient**, respectively.

The circuit can thus be modelled by the following three differential equations:

(1)

$$\dot{I} = \frac{V_0\, sin\theta - IR - V}{L}$$

$$\dot{V} = \frac{I - I_d(V)}{C(V)}$$

$$\dot{\theta} = \omega, \qquad 0 \leq \theta \leq 2\pi$$

As you can see, the system has three coupled, dynamical variables and thus has the potential for chaos. Indeed, for certain values of the circuit parameters, both the model and the real circuit do exhibit chaotic behaviour. I like to call the chaotic attractor for this system the “varactor attractor”!

Many experiments have been done to look at the circuit’s behaviour as the parameters are varied. These results can easily be compared to the model by solving the above equations using numerical integration techniques. In practice, it is easiest to fix all the parameters and simply vary V_0, the driving voltage amplitude.

Since it is generally more difficult to analyse behaviour in 3d, physicists tend to look for ways to effectively ‘reduce’ the dimensionality of a system. One way to do this is to plot a phase space projection, i.e. a plot in 2d of only two of the variables. This provides a side-on view of the system’s attractor. However, a more useful way of viewing the motion of the system in two dimensions is the **Poincaré section**. This is well suited to our varactor system since the phase of the driving voltage, θ, is periodic (with period 2π).

To form the Poincaré section then, we plot the values of I and V at regular intervals of 2π. This is equivalent to ‘cutting’ the attractor with the plane θ = constant and marking a point whenever the trajectory intersects the plane.

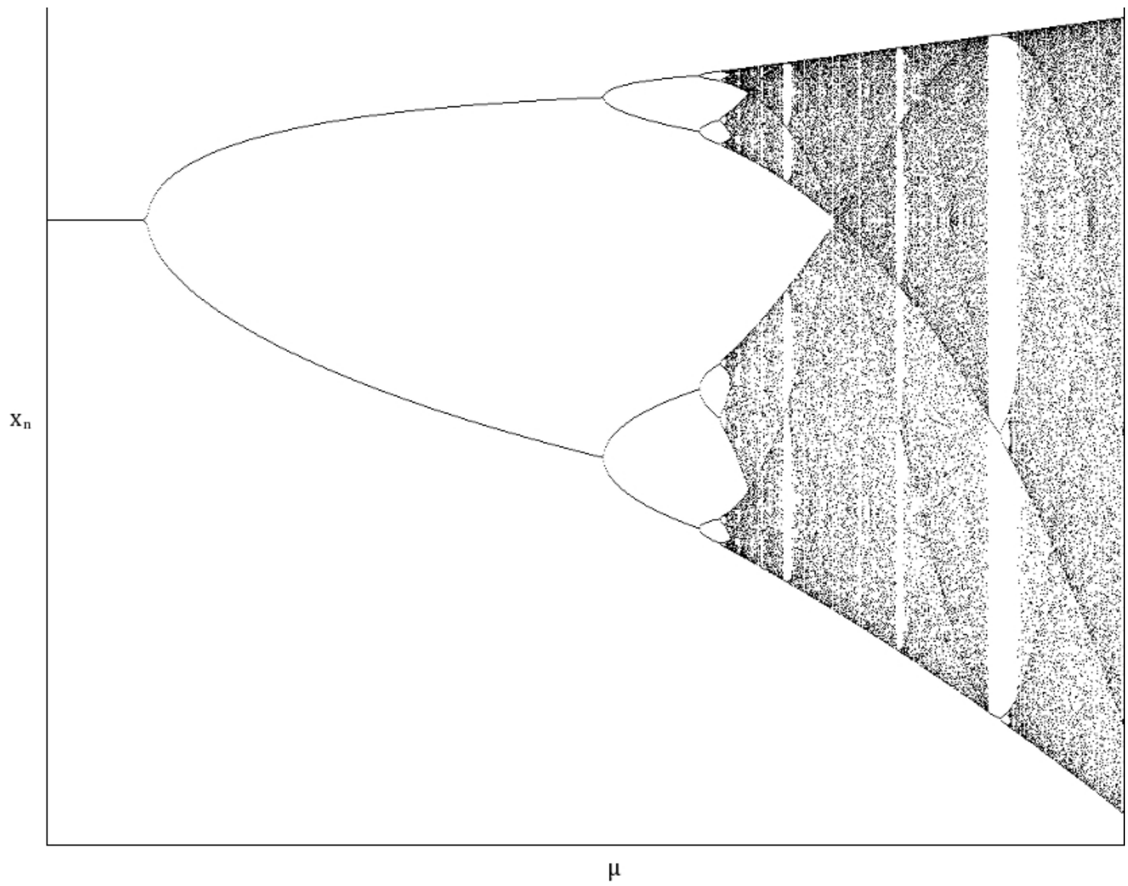
The assumed uniqueness and determinism of the solution to differential equations imply the existence of a Poincaré map function which maps one point on the Poincaré plane to the next. In other words, there exists a pair of functions which map V_n to V_{n+1} and I_n to I_{n+1}:

$$V_{n+1} = P_V(V_n, I_n)$$

$$I_{n+1} = P_I(V_n, I_n)$$

Interestingly, Poincaré map functions often reduce to a one-dimensional map. We can recognise this 1d behaviour by recording measured values of some variable *x* taken at successive intervals of time. If we plot x_{n+1} as a function of x_n and find the graph results in a single-valued functional relationship, then we say that the evolution is essentially 1-dimensional. This is called a **return mapping**. When the varactor circuit is highly dissipative, i.e. when it has a low Q factor, it may be reduced to 1-dimension in this way. This turns out to be the most useful way of characterising the dynamics of the system. We can plot a graph of the steady-state behaviour of V or I as a function of V_0.

Experimentally produced graphs showing steady-state values of I versus V_0 look remarkably similar to the following diagram produced using a quadratic mapping known as the **Logistic map**:



What we see revealed here is the period-doubling route to chaos, which is a feature of many systems. This means that, for some value of the parameter being varied, there is a steady state or period 1 behaviour, i.e. the I_n converge to some fixed value. Then, as the parameter is varied, there is a **bifurcation**, i.e. the behaviour changes to period 2 in which I_n alternates between two values.

These bifurcations continue through periods 4, 8, 16, … ∞ until chaos is reached. The dark bands are chaotic, in that I_n visits infinitely many points.

The Logistic map is given by the equation:

$$x_{n+1} = \mu x_n(1 - x_n)$$

Its return mapping is a simple parabola, but what it has in common with our experimental return mapping is the existence of a single maximum. Mitchell Feigenbaum proved that for all such maps there are bifurcation diagrams which share the same scaling ratios. This is an example of the universality of chaos. The actual system doesn't matter, just the nature of the maximum (or minimum) of its return mapping.

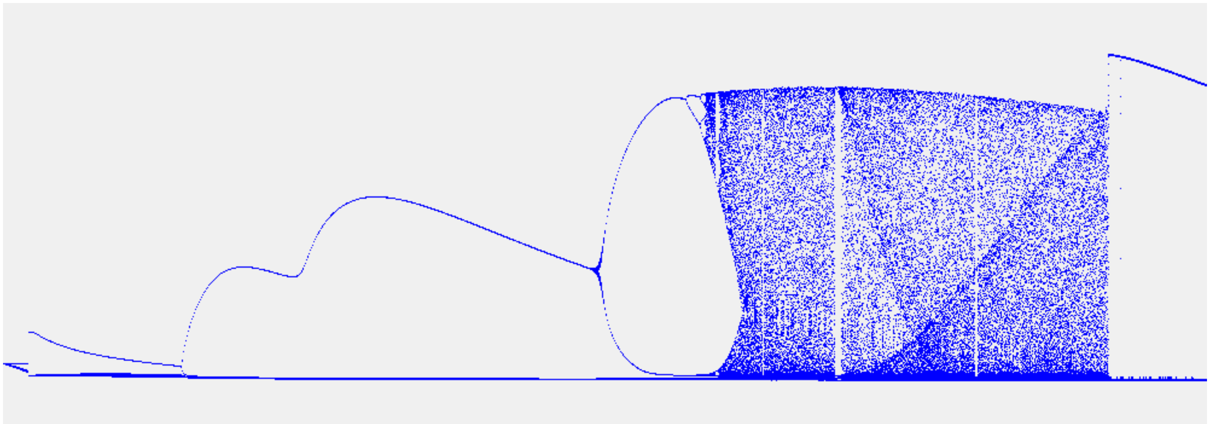
Conclusion

To conclude, I wish to quote Enrico Fermi: "It does not say in the Bible that all laws of Nature are expressible linearly!" (Gleick, 1988, p.68)

Nonlinearity in Nature is actually the rule, not the exception. We have seen that determinism does not necessarily imply that a system will behave predictably. In fact, thanks to Heisenberg, we now know that it is impossible, even theoretically, to predict the long-term behaviour of a system, since its initial conditions cannot be known to infinite precision. To put it another way: modern science has put to death Laplace's clockwork vision of the universe.

Appendix A

Bifurcation diagram of varactor voltage (V) plotted against peak driving voltage (V_0), as V_0 is varied between 0V and 5V.



The diagram was produced using the Runge-Kutta method to integrate (1) with a time interval of 5ns. After allowing 500 cycles for the system to reach its steady-state behaviour, the following 500 iterations were plotted on the diagram. The circuit parameters were taken as follows:

R = 90Ω, L = 0.01H, γ = 0.5, Φ = 0.6V, C_{j0} = 0.6nF, C_{s0} = 0.6pF, I = 4.8nA, T = 464.3K

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