Re-examining radioactive decay: a rigorous approach

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Abstract

The radioactive decay constant is prone to a conceptual misunderstanding. This article addresses the issue by providing a rigorous treatment of the physics and mathematics of radioactive decay. The pitfalls of using the decay constant are illustrated and an alternative approach proposed in which probability is emphasised and the classic ‘radioactive dice’ experiment plays a central role.

Introduction

The Rutherford-Soddy decay equation for the number of radioactive nuclei (N_t) remaining in a sample after some time t is conventionally presented in Rutherford’s original form (Rutherford 1904 p.189):

$$N_t = N_0 \, \mathrm{e}^{-\lambda t}$$

where λ is the radioactive decay constant. Rutherford concluded that: “λ has thus a distinct physical meaning, and may be defined as the proportion of the total number of systems present which change per second” (1904 p.190). In this article I will argue that Rutherford's definition of λ was erroneous, leading to a misunderstanding that has persisted to the present day.

The decay equation

When discussing the decay equation, it is important to remember that it is a statistical model which works well for very large numbers of nuclei, but does not *truly* describe their decay. Students should realize that the atomic nature of matter implies that radioactive decay is actually a discrete process (as is, for example, the rolling of dice) and that the continuous curve of the decay equation is an approximation.

The next point to make is that the form of the decay equation is simply one of convention. It can of course be written using an exponential function of any base. To a student, base e may not seem a natural choice. In fact, we will see later that it is constructive to consider an alternative form.

An ingrained misconception

Johnson *et al* (2000) define λ as “the probability that an individual nucleus will decay within a unit of time” (p.342). They suggest that, given a radionuclide with $\lambda = 0.2\ \mathrm{s}^{-1}$, “In a sample of 1000 nuclei, 200 will have decayed after 1 second”. To check this, we put the numbers into the decay equation:

$$\begin{aligned} Decayed\ nuclei &= N_0 - N_t = N_0 - N_0 \, \mathrm{e}^{-\lambda t} \\ &= 1000\,(1 - \mathrm{e}^{-0.2}) = 181 \\ &\neq 200 \end{aligned}$$

Clearly, Johnson *et al* are mistaken, but they are not alone. Authors from Eisberg (1961 p.612) through to Dobson *et al* (2002) and even Teaching Advanced Physics [n.d.] closely follow Rutherford’s definition of λ, referring to it as a probability per second of a nucleus decaying. That this definition is incorrect is readily seen by considering a short-lived radioisotope such as Ni55 which has a half-life of 212.1 ms (Firestone 1999) and hence a decay constant of 3.268 s^{-1}. Obviously this cannot be a probability per second as it is larger than 1. In fact, λ is not a probability at all and it is this single misunderstanding that leads to problems such as the above example and the half-life discrepancy discussed by Murray and Hart (2012).

The importance of probability

Probability is a key feature of radioactive decay – a consequence of its quantum nature – and as such warrants a proper treatment. It is common to introduce the decay constant as a probability divided by time: a less than intuitive concept which is of little use given the non-linear nature of the decay equation. It is far more instructive to take an alternative approach in which P_t, the probability of a nucleus decaying during some time *t*, is introduced instead. This is equal to the proportion of nuclei that decay during that time, providing the number of nuclei involved is large enough for the decay equation to hold. P_t is related to λ as follows:

$$P_t = -\frac{N_t - N_0}{N_0} = 1 - \frac{N_0\,\mathrm{e}^{-\lambda t}}{N_0} = 1 - \mathrm{e}^{-\lambda t} \neq \lambda t$$

However, in the limit of very small λ*t*:

$$P_t \approx \lambda t$$

(see Appendix), which is where the confusion arises. This misleading approximation was introduced by Rutherford (1904 p.187) who seemed to later forget it when assigning λ its "distinct physical meaning". Rutherford's mistake was a conceptual one. It is important to realize that it is P_t *not* λ*t* that is the probability of a nucleus decaying in time *t*. Imagine, for example, a radionuclide with a half-life of 1 second. In this case, the probability of each nucleus decaying in 1 second is 1/2, but its decay constant is ln(2) s^{-1} not 1/2 s^{-1}.

Demystifying the decay constant

If λ is not the probability of a nucleus decaying per unit of time, just what is it? Does it have any physical significance?

Differentiating the decay equation with respect to time we see that:

$$-\frac{dN_t}{dt} = \lambda N_0 \mathrm{e}^{-\lambda t} = \lambda N_t$$

$-\mathrm{d}N_t/\mathrm{d}t$ is the rate of decay, i.e. the activity, which is proportional to the number of nuclei remaining at any given instant. λ is simply the constant of proportionality in this relationship. However, authors such as Duncan (1975) rearrange the above equation to express λ as:

$$\lambda = -\left(\frac{dN_t}{dt}\right)/N_t = -\frac{dN_t}{N_t \cdot dt}$$

This is taken to imply that λ is equal to the fractional change in N_t per unit of time (Duncan 1975 p.488) – but that conclusion is mathematically flawed. As Landauer (1961) points out: $\mathrm{d}N_t/N_t$ "cannot be treated as an ordinary arithmetic fraction".

It would seem that, contrary to Rutherford's assertion, λ cannot be given a simple physical definition.

'Radioactive dice'

The 'radioactive dice' experiment is an excellent analogy for radioactive decay on many levels and is the perfect opportunity to give students a hands-on feel for the probabilistic nature of the quantum universe (although one might point out that the pseudo-random nature of dice is due to their classical dependence on initial conditions, whereas the quantum mechanics underlying radioactivity is truly random).

In the experiment, a unit of time is represented by a mass roll of all the 'undecayed' dice 'nuclei'. Typically one face is chosen to represent a decay, hence giving a probability of 1/6 for a six-sided die to 'decay' within the chosen unit of time. With enough dice, the data from this experiment matches the theoretical model rather well, as long as one remembers that λ is not equal to 1/6 per unit of time, as is commonly supposed. 1/6 is the probability, not the decay constant. It is this oversight that leads to Murray and Hart's half-life discrepancy (2012). To illustrate this point consider the number of dice left after *n* rolls:

$$N_n = N_0 \, (1 - P_1)^n$$

where P_1 is the probability of a die 'decaying' on each roll (i.e. 1/6 for six-sided dice). Since one roll of the dice represents one unit of time, we can clearly model this as:

$$N_t = N_0 \, (1 - P_1)^{t/t_1}$$

where t_1 is our unit of time. Starting from this form of the decay equation, the Rutherford-Soddy form can be derived using the properties of logarithms:

$$\begin{aligned} N_t = N_0 \, &\exp[\ln(1 - P_1)^{t/t_1}] \\ &= N_0 \exp\,[(t/t_1) \cdot \ln(1 - P_1)] \\ &= N_0 \, \mathrm{e}^{-\lambda t} \end{aligned}$$

where

$$\lambda = -\ln(1 - P_1)\,/t_1$$

Hence, when P_1 = 1/6, λ = 0.1823 per unit of time.

It is also worth noting that although the dice are rolled together in discrete turns, they do not actually all land simultaneously; in the same way that not all nuclei decay simultaneously during any one unit of time. A more important observation is that by using a finite set of dice, the random nature of decay becomes apparent. We see that decay does not follow the mathematical model exactly. It should become apparent that the more dice thrown, the closer the experiment matches the statistical model; whereas when there are fewer dice, the 'decay' fluctuates far more randomly. This again matches the behaviour of radioactive nuclei.

Conclusion

We have seen that the decay constant is not the probability of a nucleus decaying per unit time. Even if we follow authors such as Azzopardi and Stewart (1995) and specify that the unit of time chosen must be smaller than the radionuclide's half-life, this "definition" is still at best a numerical approximation (see Appendix). Because of this widespread misinterpretation, I recommend that emphasis instead be placed on the probability of a nucleus decaying in unit time (P_1) and the half-life ($t_{1/2}$). These more fundamental concepts can be demonstrated easily with minimal mathematics using the 'radioactive dice' experiment. If the decay constant must be introduced, I suggest it be treated purely as a convenient mathematical constant, following the treatments of Breithaupt (2000) and Halliday *et al* (2005).

Appendix

Although the decay constant is not the probability of a nucleus decaying per unit time, it turns out to be *approximately* true when λ is small enough. This may be shown as follows. For all real x, the exponential function can be expressed as an infinite power series (Boas 1983 p.24):

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

From this series expansion, we see that for sufficiently small $x < 1$:

$$e^x \approx 1 + x$$

So, for small enough λ, we can write the probability of a nucleus decaying in unit time as:

$$P_1 = 1 - e^{-\lambda t_1} \approx \lambda t_1$$

Figure 1 shows that this is only accurate (up to 2 decimal places) for $\lambda t_1 < 0.10$, which means that the approximation of λ being equal to the probability of decay per unit time only holds true for radionuclides with a half-life greater than 6.9 units of time, since:

$$t_{1/2} = \frac{\ln(2)}{\lambda} = \frac{0.6931}{0.10/t_1} = 6.9\ t_1$$

Because λ has the dimensions of $[\text{time}]^{-1}$, it follows that the smaller we choose our unit of time, the smaller the magnitude of λ. Hence it is true to say that the decay constant approximately equals the probability of a nucleus decaying per unit time only if we choose to measure time in small enough units relative to the radioisotope's half-life. This is unnecessary however, as it is trivial to calculate P_1 from λ or λ from P_1.

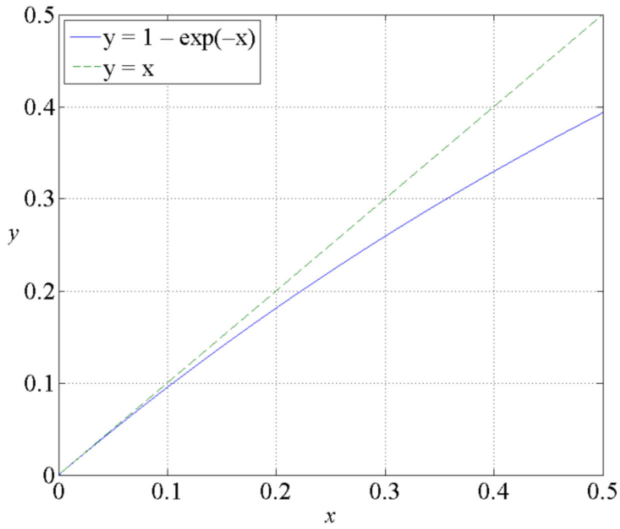

Figure 1. Graph comparing $y = 1 - e^{-x}$ to $y = x$

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