Sliding Object on a Frictionless, Rotating Ramp

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Introduction

An analysis based on an interesting 1st year undergraduate question on Classical Mechanics regarding an object at rest "on a long, frictionless, inclined plane", which "starts at an angle $\theta = 0$ and is raised such that the angle increases linearly with time i.e. $\theta = kt$ where *k* is a constant with units of rad s^{-1}". The whole system is assumed to be "in a uniform gravitational field, *g*, directed downwards" [1].

To simplify the problem, we will model the object on the plane as a dimensionless particle of mass *m*. We will consider the object to be sliding rather than rolling. We will assume that the plane is long enough to ensure the particle never reaches the bottom and that the plane is pivoted at the origin. We will also start by assuming that the particle does not leave the surface of the plane. In the final section, we will derive an expression for the conditions necessary for this to hold, but for the first section we will simply assume it to be true.

The only forces acting on the particle, then, are its weight and the normal contact force due to the plane. This is illustrated in Figure 1. If we split these forces into components perpendicular and parallel to the plane, we find from Newton's 2nd Law that:

$$ma_{\perp} = N - mg\ cos\theta$$

$$ma_{\parallel} = -mg\ sin\theta \Longrightarrow a_{\parallel} = -g\ sin\theta = -g\ sin(kt)$$

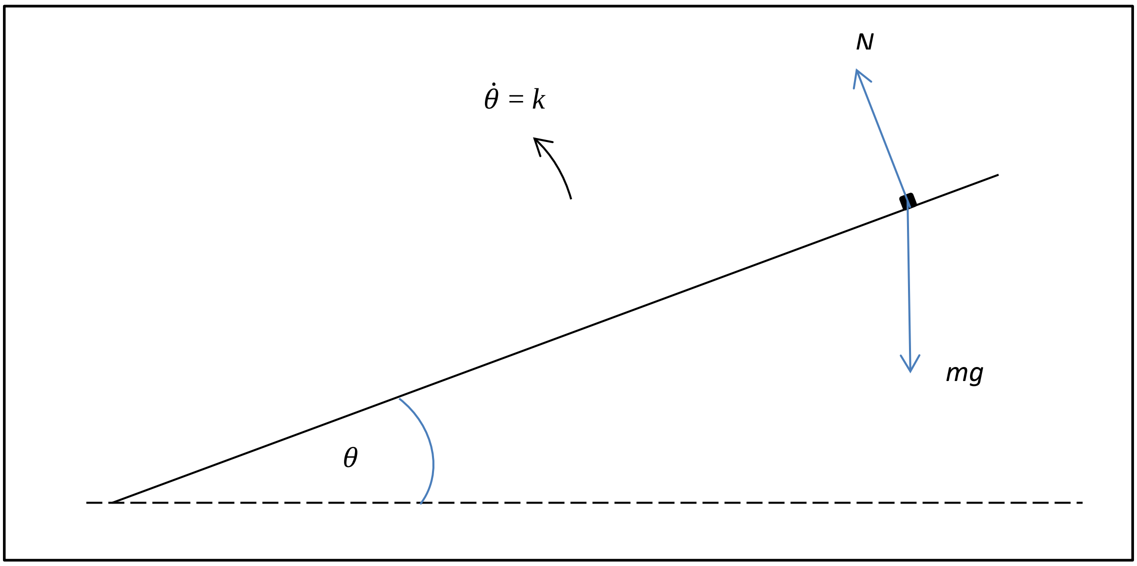

Figure 1: Free body diagram

[1] Tymms, V. First Year Assessed Problem Sheet 1 [26th October 2012] Imperial College London, Physics Department. p.2

Deriving the equation of motion

In a problem of this nature, it is most convenient to express the position, velocity and acceleration of the object in plane polar coordinates using unit base vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$, where:

$$\hat{\boldsymbol{r}} = cos\theta\ \boldsymbol{i} + sin\theta\ \boldsymbol{j}$$

$$\hat{\boldsymbol{\theta}} = -sin\theta\ \boldsymbol{i} + cos\theta\ \boldsymbol{j}$$

Then the particle's position is given by:

$$\boldsymbol{r} = r\,\hat{\boldsymbol{r}}$$

where *r* is the displacement of the particle along the plane.

By differentiating this expression, we have the particle's velocity as:

$$\dot{\boldsymbol{r}} = \dot{r}\,\hat{\boldsymbol{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}$$

Differentiating again gives the particle's acceleration as:

$$\ddot{\boldsymbol{r}} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\boldsymbol{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\boldsymbol{\theta}}$$ [2]

The acceleration components parallel and perpendicular to the plane are therefore expressed in polar coordinates by:

$$a_{\parallel} = (\ddot{r} - r\dot{\theta}^2)$$

$$a_{\perp} = (2\dot{r}\dot{\theta} + r\ddot{\theta})$$

Since $\theta = kt$, our expression for $a_{\parallel}$ can be written as:

$$a_{\parallel} = (\ddot{r} - rk^2) \quad (1)$$

We now have a simple differential equation, which we can solve to find *r* as a function of *t*:

$$a_{\parallel} = (\ddot{r} - rk^2) = -g\ sin\theta = -g\sin(kt)$$

i.e.:

$$\ddot{r} - rk^2 + g\sin(kt) = 0 \quad (2)$$

[2] For derivation, see: Bostock, L. & Chandler, S. Applied Mathematics 2, Cheltenham: Stanley Thornes (1976) p. 87

The solution to this second-order linear differential equation can be shown to be:

$$r = Ae^{kt} + Be^{-kt} + \left(\frac{g}{2\,k^2}\right)\sin(kt) \quad (3)$$

where A and B are constants of integration.

To evaluate these constants, we use our initial conditions. At time $t = 0$, $r = r_0$ and $\dot{r} = 0$.

Then:

$$r_0 = A + B \Rightarrow A = r_0 - B$$

$$\dot{r} = Ake^{kt} - Bke^{-kt} + \left(\frac{g}{2k}\right)\cos(kt)$$

$$0 = A - B + \left(\frac{g}{2k^2}\right) \Rightarrow A = B - \frac{g}{2k^2}$$

$$B = \frac{r_0}{2} + \frac{g}{4k^2}$$

$$A = \frac{r_0}{2} - \frac{g}{4k^2}$$

Thus, our solution becomes:

$$r = \left(\frac{r_0}{2} - \frac{g}{4k^2}\right)e^{kt} + \left(\frac{r_0}{2} + \frac{g}{4k^2}\right)e^{-kt} + \left(\frac{g}{2\,k^2}\right)\sin(kt)$$

$$\Rightarrow r = \frac{r_0}{2}\left(e^{kt} + e^{-kt}\right) + \frac{g}{4k^2}\left(e^{-kt} - e^{kt}\right) + \frac{g}{2\,k^2}\sin(kt)$$

$$\Rightarrow r = r_0\cosh(kt) - \frac{g}{2k^2}\sinh(kt) + \frac{g}{2\,k^2}\sin(kt)$$

$$\Rightarrow r = r_0\cosh(kt) + \frac{g}{2\,k^2}\left(\sin(kt) - \sinh(kt)\right) \quad (4)$$

Condition for the object to leave the ramp

We will now consider under what condition the object leaves the surface of the ramp. When this happens, the equation of motion we derived above will become invalid. Intuitively, we would expect that when the normal contact force changes from positive to negative, the object must have left the ramp.

Applying Newton's 2nd Law to the forces perpendicular to the plane gives:

$$N - mg\ cos\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta})$$

$$\Rightarrow N = m(2\dot{r}\dot{\theta} + r\ddot{\theta}) + mg\ cos\theta$$

Since $\theta = kt$, we have: $\dot{\theta} = k$ and $\ddot{\theta} = 0$. Also, at time $t = 0, \theta = 0$

$$\therefore N = m(2\dot{r}k) + mg\cos(kt)$$

But $\dot{r} = kr_0 \sinh(kt) + \frac{g}{2\,k}(\cos(kt) - \cosh(kt))$

$$\Rightarrow N = mk\left(2kr_0 \sinh(kt) + \frac{g}{k}(\cos(kt) - \cosh(kt))\right) + mgcos(kt)$$

$$\Rightarrow N = m\big(2k^2 r_0 \sinh(kt) - g\cosh(kt) + 2g\ cos(kt)\big) \quad (5)$$

Thus, the particle leaves the ramp when:

$$2k^2 r_0 \sinh(kt) - g\cosh(kt) + 2g\ cos(kt) = 0 \quad (6)$$

providing:

$$2k^2 r_0 \sinh(k(t + \delta t)) - g\cosh\big(k(t + \delta t)\big) + 2g\ cos\big(k(t + \delta t)\big) < 0$$

where δt represents a small change in t.